Monotonicity of multi-point Padé approximants

A. Bultheel *

Department of Computer Science, K.U.Leuven, Belgium
E-mail: Adhemar.Bultheel@cs.kuleuven.ac.be

P. González-Vera **

Department of Mathematical Analysis, Univ. La Laguna, Tenerife, Spain
E-mail: pglez@ull.es

E. Hendriksen

Department of Mathematics, University of Amsterdam, The Netherlands.
E-mail: erik@wins.uva.nl

O. Njåstad

Department of Math. Sc., Norwegian Univ. of Science and Technology, Trondheim, Norway
E-mail: njastad@math.ntnu.no

The main purpose of this paper is to discuss behavior of multi-point Padé approximants of the Stieltjes transform of measures which solve a certain rational moment problem. The location of the interpolation points and the support of the measures satisfy conditions which make the problem analogous to the classical and strong (two-point) Stieltjes moment problems. The main result is that on a certain interval outside the support of the measures, the sequences of even and of odd $[n-1/n]$ multi-point Padé approximants converge *monotonically* to Stieltjes transforms of solutions of the moment problem.

Keywords: Orthogonal rational functions, Multi-point Padé approximants, Rational moment problems.

AMS Subject classification: 30E05, 40A15, 41C30, 42C05

1. Introduction

By the *Stieltjes transform* of a finite positive measure μ on the real line $(-\infty, \infty)$ we shall here mean the function

$$S(z, \mu) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{z-t}. \tag{1.1}$$

It is well known that if μ has support in the non-negative real line $[0, \infty)$ then the $[n-1/n]$ Padé approximants of $S(z, \mu)$ at $z = \infty$ form a decreasing sequence for $z \in (-\infty, 0)$, with limit equal to $S(z, \nu_0)$ for a certain solution ν_0 of the moment problem assosiated with μ. Certain related rational

* The work of the first author is partially supported by the Fund for Scientific Research (FWO), project "Orthogonal systems and their applications", grant #G.0278.97 and the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with the authors.

** The work of this author is partially supported by the scientific research project PB96-1029.

approximants form an increasing sequence for $z \in (-\infty, 0)$, with limit equal to $S(z, \nu_\infty)$ for a certain solution ν_∞ of the moment problem. See Section 2.

If μ has support in $[0, \infty)$, a sequence of two-point Padé $[n-1/n]$ approximants of $S(z, \mu)$ at ∞ and 0 has the following property: The subsequence with even indices forms a decreasing sequence for $z \in (-\infty, 0)$, with limit equal to $S(z, \mu_0)$ for a certain solution μ_0 of the strong moment problem associated with μ, while the subsequence with odd indices forms an increasing sequence for $z \in (-\infty, 0)$, with limit equal to $S(z, \mu_\infty)$ for a certain solution μ_∞ of the strong moment problem. See Section 3.

The main purpose of this paper is to discuss a general situation where the interpolation point at ∞ and 0 are replaced by a sequence $\{\alpha_n\}$ of points on the real line. These points are assumed to satisfy the condition $\alpha_{2m} \leq \alpha < \beta \leq \alpha_{2m+1} \leq \gamma$, and μ is assumed to have support in $[\gamma, \infty)$. Similar results as those above are valid for $z \in (\alpha, \beta)$ for the even and odd subsequences of a sequence of $[n-1/n]$ multi-point Padé approximants of $S(z, \mu)$.

In Sections 2 and 3 we sketch the (classical) one-point and the (strong) two-point situation. In Section 4 we describe the extended problem and introduce the corresponding orthogonal and associated orthogonal functions and their quotients. In Section 5 we establish that these quotients are multi-point Padé approximants of $S(z, \mu)$. Finally in Section 6 we state the convergence properties of these multi-point Padé approximants.

In order to distinguish notationally between the three situations considered, some of the notation used here is not consistent with notation used in earlier contributions.

2. The classical Stieltjes problem

Let μ be a positive measure on $[0, \infty)$ such that the moments

$$c_n = \int_0^\infty t^n \mathrm{d}\mu(t), \quad n = 0, 1, 2, \ldots \tag{2.1}$$

exist. A measure ν on $[0, \infty)$ for which $c_n = \int_0^\infty t^n \mathrm{d}\nu(t)$ for $n = 0, 1, 2, \ldots$, is called a solution of the *Stieltjes moment problem (SMP)* determined by $\{c_n\}_{n=0}^\infty$.

Let $\{Q_n\}$ be the system of monic orthogonal polynomials corresponding to the basis $\{1, z, z^2, \ldots\}$ for the space Π of polynomials. Orthogonality is with respect to μ or, wath is the same, with respect to any solution of the SMP. The associated orthogonal polynomials $\{P_n\}$ are defined by

$$P_n(z) = \int_0^\infty \frac{Q_n(t) - Q_n(z)}{t - z} \mathrm{d}\mu(t), \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

The functions $\{P_n, Q_n\}$ satisfy a three-term recurrence relation of the form

$$\begin{bmatrix} P_n(z) \\ Q_n(x) \end{bmatrix} = (z - a_n) \begin{bmatrix} P_{n-1}(z) \\ Q_{n-1}(z) \end{bmatrix} + \lambda_n \begin{bmatrix} P_{n-2}(z) \\ Q_{n-2}(z) \end{bmatrix}, \quad n = 1, 2, \ldots \tag{2.3}$$

$$\begin{bmatrix} P_0 \\ Q_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} P_{-1} \\ Q_{-1} \end{bmatrix} = \begin{bmatrix} c_0 \\ 0 \end{bmatrix}, \tag{2.4}$$

with

$$a_1 > 0, \ \ \lambda_1 > 0, \ \ a_n > 0, \ \ \lambda_n < 0 \ \text{ for } \ n = 2, 3, \ldots . \tag{2.5}$$

(The constants P_{-1} and λ_1 are not uniquely determined, but it is required that $\lambda_1 P_{-1} = c_0$). See e.g. [1, p. 232], [13, p. 18-19], [32, p. 231].

The quotients $P_n(z)/Q_n(z)$ are thus approximants of a continued fraction associated with (2.3)-(2.5). This continued fraction is a *G-fraction (Grommer fraction)* or *real J-fraction (real Jacobi fraction)*. See e.g. [32, p. 192], [35, p. 3], [37, p. 6].

The rational function $P_n(z)/Q_n(z)$ is the $[n-1/n]$ Padé approximant at ∞ of the function $S(z, \mu)$ or equivalently of the series $\sum_{n=0}^{\infty} c_n z^{-(n+1)}$, in the following sense:

$$\frac{P_n(z)}{Q_n(z)} - S(z, \mu) = O\left(\frac{1}{z^{2n+1}}\right) \ \text{ at } \ z = \infty. \tag{2.6}$$

See e.g. [1, p. 22-23], [22, p. 45], [24, p. 383].

The following result is well known. See e.g. [22, p. 46], [28, p. 39], [32, p. 229], [33, p. 73-74], cf. also [2, p. 215], [3, p. 170].

Theorem 2.1. *For $x \in (-\infty, 0)$ the sequence $P_n(x)/Q_n(x)$ is decreasing to $S(z, \nu_0)$, where ν_0 is a certain solution of the SMP.*

The moment problem determined by the sequence $\{c_n\}$ also gives rise to an *S-fraction (Stieltjes fraction)*. The canonical numerators $\{A_n\}$ and denominators $\{B_n\}$ satisfy the following recursion:

$$\begin{bmatrix} A_{2m}(z) \\ B_{2m}(z) \end{bmatrix} = \begin{bmatrix} A_{2m-1}(z) \\ B_{2m-1}(z) \end{bmatrix} + b_{2m} \begin{bmatrix} A_{2m-2}(z) \\ B_{2m-2}(z) \end{bmatrix}, \ \ m = 1, 2, \ldots \tag{2.7}$$

$$\begin{bmatrix} A_{2m+1}(z) \\ B_{2m+1}(z) \end{bmatrix} = z \begin{bmatrix} A_{2m}(z) \\ B_{2m}(z) \end{bmatrix} + b_{2m+1} \begin{bmatrix} A_{2m-1}(z) \\ B_{2m-1}(z) \end{bmatrix}, \ \ m = 0, 1, 2, \ldots \tag{2.8}$$

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \begin{bmatrix} A_{-1} \\ B_{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{2.9}$$

with

$$b_n > 0 \text{ for } n = 1, 2, \ldots . \tag{2.10}$$

The even contraction of this S-fraction is the G-fraction (2.3)- (2.5). See [24, p. 357], [32, p. 214-220], [33, p. 73], [35, p. 3], [37, p. 6]. More precisely we have

$$A_{2n}(z) = P_n(z), \ \ B_{2n}(z) = Q_n(z), \ \ n = 0, 1, 2, \ldots , \tag{2.11}$$

(note that the polynomials B_n are monic), and thus $P_n(z)/Q_n(z) = A_{2n}(z)/B_{2n}(z)$.

We set

$$R_n(z) = A_{2n+1}(z), \ \ T_n(z) = B_{2n+1}(z), \ \ n = 0, 1, 2, \ldots \tag{2.12}$$

Then $\{R_n(z)/T_n(z)\}$ are the approximants of the odd contraction of the S-fraction (2.7)-(2.10). Here $zR_n(z)/T_n(z)$ is the $[n/n]$ Padé approximant of $z\,S(z,\mu)$. This follows e.g. from [24, p. 382] by the substitution $z \to 1/z$. (Setting $F(z,\mu) = z\,S(z,\mu)$ and $F_n(z) = zR_n(z)/T_n(z)$, one has $F(z,\mu) - F_n(z,\mu) = O(z^{-(2n+1)})$ according to the definition of the $[n/n]$ Padé approximant at ∞. Thus $R_n(z)/T_n(z) - S(z,\mu) = O(z^{-2n})$, and $R_n(z)/T_n(z)$ is not the $[n-1/n]$ Padé approximant of $S(z,\mu)$ at $z=\infty$, one interpolation condition missing.)

The following result is also well known. See e.g. [28, p. 45-48], [32, p. 214-220], [33, p. 73-74], cf. also [22, p. 46].

Theorem 2.2. *For $x \in (-\infty,0)$ the sequence $\{R_n(z)/T_n(z)\}$ is increasing to $S(z,\nu_\infty)$, where ν_∞ is a certain solution of the SMP.*

Furthermore the following result holds. See e.g. [28, p. 45], cf. also [1, p. 237-239].

Theorem 2.3. *For every solution ν of the SMP we have*

$$S(x,\nu_\infty) \le S(x,\nu) \le S(x,\nu_0) \tag{2.13}$$

when $x \in (-\infty,0)$.

A moment problem is called *determinate* if it has a unique solution. From Theorem 2.3 immediately follows:

Corollary 2.4. *The SMP is determinate if and only if $S(x,\nu_0) = S(x,\nu_\infty)$ for all $x \in (-\infty,0)$*

In other words: The SMP is determinate if and only if the continued fraction (S-fraction) (2.7)-(2.9) associated with the moment problem converges for $z \in (-\infty,0)$.

Remark 2.5. While the even approximants $A_{2n}(z)/B_{2n}(z)$ of the S-fraction (2.7)-(2.10) equal the approximants $P_n(z)/Q_n(z)$ of the G-fraction (2.3)-(2.5), the odd approximants $A_{2n+1}(z)/B_{2n+1}(z)$ equal certain *quasi-approximants* of the G-fraction. This means that

$$\frac{A_{2n+1}(z)}{B_{2n+1}(z)} = \frac{R_n(z)}{T_n(z)} = \frac{P_n(z)+\tau_n P_{n-1}(z)}{Q_n(z)+\tau_n Q_{n-1}(z)} \tag{2.14}$$

for a certain value of the parameter τ_n. See [1, p. 232-238], [28, p. 48].

3. The strong Stieltjes moment problem

Let μ be a positve measure on $[0,\infty)$ such that the moments

$$c_n = \int_0^\infty t^n \mathrm{d}\mu(t), \quad n = 0, \pm 1, \pm 2, \ldots \tag{3.1}$$

exist. A measure ν on $[0,\infty)$ for which $c_n = \int_0^\infty t^n \mathrm{d}\nu(t)$ for $n = 0, \pm1, \pm2, \ldots$, is called a solution of the *strong Stieltjes moment problem (SSMP)* determined by $\{c_n\}_{n=-\infty}^{\infty}$. For basic information on the SSMP, see e.g. [21], [29], [30].

By orthonormalization of the basis $\{1, z^{-1}, z, z^{-2}, z^2, \ldots\}$ for the space Λ of Laurent polynomials (finite linear combinations of monomials z^n, $n = 0, \pm1, \pm2, \ldots$) we obtain a μ-orthonormal system

$\{\rho_n\}$ of Laurent polynomials. Orthogonality is with respect to μ, or equivalently, with respect to any solution of the SSMP. The associated orthogonal Laurent polynomials $\{\pi_n\}$ are defined by

$$\pi_n(z) = \int_0^\infty \frac{\rho_n(t) - \rho_n(z)}{t - z} \mathrm{d}\mu(t), \quad n = 0, \pm 1, \pm 2, \ldots \tag{3.2}$$

The functions $\{\pi_n, \rho_n\}$ satisfy a three-term recurrence relation of the form

$$\begin{bmatrix} \pi_n(x) \\ \rho_n(z) \end{bmatrix} = (g_n + h_n z^{(-1)^n}) \begin{bmatrix} \pi_{n-1}(z) \\ \rho_{n-1}(x) \end{bmatrix} + f_n \begin{bmatrix} \pi_{n-2}(z) \\ \rho_{n-2}(z) \end{bmatrix}, \quad n = 1, 2, \ldots \tag{3.3}$$

$$\begin{bmatrix} \pi_0 \\ \rho_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{1}{\sqrt{c_0}} \end{bmatrix}, \begin{bmatrix} \pi_{-1} \\ \rho_{-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{z} \\ 0 \end{bmatrix}, \tag{3.4}$$

with

$$f_n < 0, \; (-1)^n g_n > 0, \; (-1)^n h_n < 0. \tag{3.5}$$

See [18, p. 109-110], [29, p. 322], [30, p. 64], where various equivalent formulations are given.

Note that we here work with *normalized* orthogonal functions $\{\rho_n\}$. This is the reason for the appearance of three parameter sequences $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ (which are not independent), while for the monic functions $\{Q_n\}$ in Section 2 only the two parameter sequences $\{a_n\}$, $\{\lambda_n\}$ are used. (The initial value π_{-1} and the coefficient f_1 are not uniquely determined. It is required that $f_1 \pi_{-1} = \frac{-c_{-1}\sqrt{c_0}}{\sqrt{c_0 c_{-2} - c_{-1}^2} z}$.)

The quotiens $\pi_n(z)/\rho_n(z)$ are the approximants of the continued fraction associated with (3.3)-(3.5). This continued fraction is equivalent to a *positive T-fraction (positive Thron fraction).* See [18, p. 109-110], [21, p. 508], [24, p. 245].

The rational function $\pi_n(z)/\rho_n(z)$ is the $[n-1/n]$ two-point Padé approximant of $S(z, \mu)$ at ∞ and 0, or equivalently of the series $\sum_{n=0}^{\infty} c_n z^{-n}$ and $-\sum_{n=1}^{\infty} c_{-n} z^{n-1}$, in the following sense:

$$\frac{\pi_n(z)}{\rho_n(z)} - S(z, \mu) = O(z^n) \text{ at } z = 0 \tag{3.6}$$

$$\frac{\pi_n(z)}{\rho_n(z)} - S(z, \mu) = O(\frac{1}{z^{n+1}}) \text{ at } z = \infty. \tag{3.7}$$

See [18, p. 116], [21, p. 509-510], [31, p. 81].

For these two-point Padé approximants the following results are known. See [12, p. 70] (where other diagonal sequences are also considered), [16, p. 100], [29, p. 338], cf. also [17, p. 377], [19, p. 695-703], [36].

Theorem 3.1. *For $x \in (-\infty, 0)$ the following hold:*

A. *The sequence $\{\pi_{2m}(x)/\rho_{2m}(x)\}$ is decreasing to $S(x, \mu_0)$, where μ_0 is a certain solution of the SSMP.*

B. *The sequence $\{\pi_{2m+1}(x)/\rho_{2m+1}(x)\}$ is increasing to $S(x, \mu_\infty)$, where μ_∞ is a certain solution of the SSMP.*

C. For every solution ν of the SSMP we have

$$S(x,\mu_\infty) \le S(x,\nu) \le S(x,\mu_0). \tag{3.8}$$

For more details on two-point Padé approximation, see also [20], [25], [26], [27], [34].

From Theorem 3.1.C we immediately conclude:

Corollary 3.2. *The SSMP is determinate if and only if $S(x,\mu_0) = S(x,\mu_\infty)$ for all $x \in (-\infty, 0)$.*

In other words: The SSMP is determinate if and only if the sequence of $[n-1/n]$ two-point Padé approximants of $S(z,\mu)$ (or equivalently: the continued fraction (3.3)-(3.4) associated with the moment problem) is convergent for $z \in (-\infty, 0)$.

4. Rational Stieltjes problem

The basic general theory of orthogonal rational functions and associated moment problems can be found in [4], [6], [7], [8], [10].

Let $\{\alpha_k\}_{k=1}^\infty$ be a sequence of not necessarily distinct points on the real line $(-\infty,\infty)$. We define

$$\omega_0 = 1,\ \omega_n(z) = (z-\alpha_1)\dots(z-\alpha_n),\ \ n = 1,2,\dots. \tag{4.1}$$

The space $\mathcal{L}$ is defined to be the linear span of the sequence $\{1/\omega_0, 1/\omega_1, \dots, 1/\omega_n, \dots\}$. Thus a function belongs to $\mathcal{L}$ if and only if for some n it may be written in the form $p(z)/\omega_n(z)$, where p is a polynomial of degree at most n.

For technical reasons the posibility of α_k being zero is excluded in [10], but it is included in [6]. On the other hand, points α_k are allowed to be infinite in [10], and the polynomial situation then appears as a special case. To keep technicalities simple, we do not include this situation here.

We shall here deal with a special situation, which may be considered as corresponding to the Stieltjes situation for the classical and strong problems. A theory of the rational Stieltjes problem as described below is presented in [11]. The specific results concerning this case refered to in the following can be found there.

We assume the existence of real numbers α, β, γ with $\alpha < \beta \le \gamma$ and a positive measure μ with infinite support such that

$$\alpha_{2m} \le \alpha,\ \ \beta \le \alpha_{2m-1} \le \gamma,\ \ m = 1,2,\dots,\ \ \text{Supp}\ \mu \subset [\gamma,\infty) \tag{4.2}$$

and such that all functions in the *product space* $\mathcal{L}\cdot\mathcal{L}$ are μ-absolutely integrable. We define the *moments* $c_{m,n}$ by

$$c_{m,n} = \int_\gamma^\infty \frac{\mathrm{d}\mu(t)}{\omega_m(t)\omega_n(t)},\ \ m,n = 0,1,2,\dots. \tag{4.3}$$

For simplicity we normalize μ such that $c_{0,0} = 1$. Note that $c_{m,n} = c_{n,m}$.

A measure ν on $[\gamma,\infty)$ is said to solve the *rational Stieltjes moment problem (RSMP) on $\mathcal{L}\cdot\mathcal{L}$* determined by $\{c_{m,n}\}_{m,n=0}^\infty$ if

$$c_{m,n} = \int_\gamma^\infty \frac{\mathrm{d}\nu(t)}{\omega_m(t)\omega_n(t)},\ \ m,n = 0,1,2,\dots, \tag{4.4}$$

and to solve the *rational Stieltjes moment problem (RSMP) on* $\mathcal{L}$ determined by $\{c_{0,n}\}_{n=0}^{\infty}$ if

$$c_{0,n} = \int_{\gamma}^{\infty} \frac{\mathrm{d}\nu(t)}{\omega_n(t)}, \quad n = 0, 1, 2, \ldots. \tag{4.5}$$

A measure which solves the moment problem on $\mathcal{L} \cdot \mathcal{L}$ also solves the problem on $\mathcal{L}$, since $\mathcal{L} \subset \mathcal{L} \cdot \mathcal{L}$. Note that $\mathcal{L} = \mathcal{L} \cdot \mathcal{L}$ if every point in the sequence $\{\alpha_k\}$ occurs an infinite number of times.

The strong (or two-point) Stieltjes problem is obtained as the limiting situation when α tends to ∞ along the negative axis for the situation $\beta = \gamma = 0$, $\alpha_{2m} = \alpha$, $\alpha_{2m-1} = 0$ for all m.

Let $\{\varphi_n\}_{n=0}^{\infty}$ denote a orthonormal sequence in $\mathcal{L}$ obtained by orthonormalization of the basis $\{1/\omega_0, 1/\omega_1, \ldots, 1/\omega_n, \ldots\}$. (Orthonormalization is with respect to the inner product on $\mathcal{L}$ generated by any solution of RSMP on $\mathcal{L} \cdot \mathcal{L}$.) We use the following sign normalization:

$$\varphi_{2m}(x) > 0, \quad \varphi_{2m+1}(x) < 0 \text{ for } x \in (\alpha, \beta). \tag{4.6}$$

This determination is unambiguous, $\varphi_n(x)$ having all its zeros in (γ, ∞). (See [11, Theorem 3.1].) We may write $\varphi_n(z)$ in the form $\varphi_n(z) = p_n(z)/\omega_n(z)$, where p_n is a polynomial of exact degree n. (See [11, Theorem 3.1].)

The associated orthogonal rational functions $\{\sigma_n\}$ are defined by

$$\sigma_n(z) = \int_{\gamma}^{\infty} \frac{\varphi_n(t) - \varphi_n(z)}{t - z} \mathrm{d}\mu(t), \quad n = 0, 1, 2, \ldots. \tag{4.7}$$

We may write $\sigma_n(z)$ in the form $\sigma_n(z) = q_n(z)/w_n(z)$, where q_n is a polynomial of degree at most $n - 1$.

The functions $\{\sigma_n, \varphi_n\}$ satisfy a three-term recurrence relation of the form

$$\begin{aligned} \begin{bmatrix} \sigma_n(z) \\ \varphi_n(z) \end{bmatrix} &= \frac{U_n(z - \alpha_{n-2}) + V_n(z - \alpha_{n-1})}{z - \alpha_n} \begin{bmatrix} \sigma_{n-1}(z) \\ \varphi_{n-1}(z) \end{bmatrix} \\ &\quad + W_n \frac{z - \alpha_{n-2}}{z - \alpha_n} \begin{bmatrix} \sigma_{n-2}(z) \\ \varphi_{n-2}(z) \end{bmatrix}, \quad n = 3, 4, 5, \ldots, \end{aligned} \tag{4.8}$$

$$\begin{bmatrix} \sigma_2(z) \\ \varphi_2(z) \end{bmatrix} = \frac{U_2(z - \alpha_2) + V_2(z - \alpha_1)}{z - \alpha_2} \begin{bmatrix} \sigma_1(z) \\ \varphi_1(z) \end{bmatrix} + W_2 \frac{1}{z - \alpha_2} \begin{bmatrix} \sigma_0 \\ \varphi_0 \end{bmatrix}, \tag{4.9}$$

$$\begin{bmatrix} \sigma_1(z) \\ \varphi_1(z) \end{bmatrix} = \frac{U_1(z - \alpha_1) + V_1(z - \alpha_2)}{z - \alpha_1} \begin{bmatrix} \sigma_0 \\ \varphi_0 \end{bmatrix} + W_1 \frac{1}{z - \alpha_1} \begin{bmatrix} \sigma_{-1} \\ \varphi_{-1} \end{bmatrix}, \tag{4.10}$$

$$\begin{bmatrix} \sigma_0 \\ \varphi_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \sigma_{-1} \\ \varphi_{-1} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \tag{4.11}$$

with

$$\left. \begin{array}{l} U_n < 0 \text{ for } n = 1, 2, 3, \ldots \\ V_n > 0 \text{ for } n = 1, 3, 4, 5, \ldots, \\ W_n > 0 \text{ for } n = 1, 2, 3, \ldots. \end{array} \right\} \tag{4.12}$$

See [6, Theorem 7.1], [11, (3.8)-(3.15), (3.41)-(3.43)]. Note that we again work with *normalized* functions and use the three parameter sequences $\{U_n\}$, $\{V_n\}$, $\{W_n\}$. (The initial value σ_{-1} and the coefficient W_1 are not uniquely determined. It is required that $W_1\sigma_{-1} = -c_{0,1}/\sqrt{c_{1,1} - c_{0,1}^2}$.)

The quotients $\sigma_n(z)/\varphi_n(z)$ are the approximants of a continued fraction associated with the recurrence (4.8)-(4.12). Continued fractions of this kind are called *(positive) MP-fractions (Multi-point Padé continued fractions)*. See e.g. [9, p. 75].

5. Multi-point Padé approximants

Recall that the quotients $\sigma_n(z)/\varphi_n(z)$ are the approximants of the continued fraction associated with (4.8)-(4.12). We shall establish that under a mild extra condition on the measure μ, these rational functions also are multi-point Padé approximants of $S(z,\mu)$. Equivalently, they may be considered as multi-point Padé approximants to certain Newton series expansions of $S(z,\mu)$, see [9, p. 84-85], [14, p. 368-373]. For basic definitions and facts concerning multi-point Padé approximants, see e.g. [2], [3], [15], [23], [38].

In [10] the following functions ψ_n and Ω_μ were studied.

$$\psi_n(z) = -i\int_\gamma^\infty \frac{1+tz}{t-z}[\varphi_n(t) - \varphi_n(z)]\mathrm{d}\mu(t), \quad n = 0,1,2,\ldots \tag{5.1}$$

$$\Omega_\mu(z) = -i\int_\gamma^\infty \frac{1+tz}{t-z}\mathrm{d}\mu(t). \tag{5.2}$$

Note that we have $\psi_n(z) = r_n(z)/\omega_n(z)$, where r_n is a polynomial of degree at most n.

Proposition 5.1. *The following equalities hold for* $n = 1,2,\ldots$:

A. $$\psi_n(z) = iz\varphi_n(z) - i(1+z^2)\sigma_n(z) \tag{5.3}$$

B. $$\Omega_\mu(z) = -iz + i(1+z^2)S(z,\mu). \tag{5.4}$$

Proof. This follows by direct calculation from (1.1), (4.7) and (5.1)-(5.2), taking into account that $\int_\gamma^\infty \mathrm{d}\mu(t) = 1$ and $\int_\gamma^\infty \varphi_n(t)\mathrm{d}t = 0$ for $n = 1,2,\ldots$. □

We shall say about a function f that it is *bounded at* $z = \alpha_k$ if the following holds: f is bounded in every region given by $\epsilon < |\arg(z-\alpha_k)| < \pi - \epsilon$, $0 < |z - \alpha_k| < r$ where $r > 0$, $\epsilon > 0$. We shall also deal with boundedness at $z = \infty$, which means boundedness in every region given by $\epsilon < |\arg z| < \pi - \epsilon$, $|z| > R$, where $\epsilon > 0$, $R < \infty$.

It is shown in [10, Theorem 11.10.12] that $\psi_n(z)/\varphi_n(z)$ is an $[n/n]$ multi-point Padé approximant to $-\Omega_\mu(z)$ at the table $\{i, -i, \alpha_1, \alpha_1, \ldots, \alpha_{n-1}, \alpha_{n-1}, \alpha_n\}$ in the following sense:

$$\frac{\psi_n(z)}{\varphi_n(z)} + \Omega_\mu(z) = \Gamma_n(z)(z-i)(z+i)\omega_n(z)\omega_{n-1}(z) \tag{5.5}$$

with $\Gamma_n(z)$ continuous at $z = \pm i$ and bounded at $\alpha_1, \ldots, \alpha_n$. (The minus sign in $-\Omega_\mu(z)$ is missing in [10]. See also [5, Theorem 2.8], where the completely analogous situation with the points α_k lying on the unit circle is dealt with.)

Theorem 5.2. *Assume that $\int_\gamma^\infty |t|\mathrm{d}\mu(t) < \infty$. Then the rational function $\sigma_n(z)/\varphi_n(z)$ is an $[n-1/n]$ multi-point Padé approximant of $S(z,\mu)$ at the table $\{\infty, \alpha_1, \alpha_1, \ldots, \alpha_{n-1}, \alpha_{n-1}, \alpha_n\}$ in the following sense:*

$$\frac{\sigma_n(z)}{\varphi_n(z)} - S(z,\mu) = \Delta_n(z)\omega_n(z)\omega_{n-1}(z) \tag{5.6}$$

with $\Delta_n(z)$ bounded at $\alpha_1, \ldots, \alpha_n$,

$$\frac{\sigma_n(z)}{\varphi_n(z)} - S(z,\mu) = D_n(z) \cdot \frac{1}{z^2} \tag{5.7}$$

with $D_n(z)$ bounded at ∞.

Proof. We find from Proposition 5.1 that

$$\frac{\sigma_n(z)}{\varphi_n(z)} - S(z,\mu) = \frac{i}{1+z^2}[\frac{\psi_n(z)}{\varphi_n(z)} + \Omega_\mu(z)]. \tag{5.8}$$

This together with (5.5) shows that (5.6) is satisfied.

We also find

$$\frac{\varphi_n(t) - \varphi_n(z)}{t-z} = -\frac{1}{z}[\varphi_n(t) - \varphi_n(z)] + \frac{1}{z^2} t f_z(t) \tag{5.9}$$

with

$$f_z(t) = \frac{z}{t-z}[\varphi_n(t) - \varphi_n(z)]. \tag{5.10}$$

Since $\int_\gamma^\infty \mathrm{d}\mu(t) = 1$ and $\int_\gamma^\infty \varphi_n(t)\mathrm{d}\mu(t) = 0$ for $n = 1, 2, \ldots$, the definition (4.7) can be written as

$$\sigma_n(z) = \frac{\varphi_n(z)}{z} + \frac{1}{z^2}\int_\gamma^\infty t f_z(t)\mathrm{d}\mu(t). \tag{5.11}$$

For every fixed angular region $\epsilon < |\arg(z)| < \pi - \epsilon$, $\epsilon > 0$, there is a positive number δ such that $|t - z| \geq \delta|z|$ for z in the region. Hence $|f_z(t)| \leq 1/\delta[|\varphi_n(t)| + |\varphi_n(z)|]$, and consequently

$$\left|\int_\gamma^\infty t f_z(t)\mathrm{d}\mu(t)\right| \leq \frac{1}{\delta}\left[\int_\gamma^\infty |t||\varphi_n(t)|\mathrm{d}\mu(t) + |\varphi_n(z)|\int_\gamma^\infty |t|\mathrm{d}\mu(t)\right]. \tag{5.12}$$

The rational function $\varphi_n(z)$ has a finite limit different from zero when z tends to ∞. It follows that $\varphi_n(z)$ is bounded and bounded away from zero at ∞. Because it is also assumed that the second integral converges, the second term in the right hand side of (5.12) is bounded. For the first term, we note that $t\varphi_n(t) = q_1(t) + \sum_{k=0}^n \eta_k/\omega_k(t)$, where $q_1(t)$ is a polynomial of degree at most 1 and the η_k are constants. Thus also the first integral will converge by the assumption $\int_\gamma^\infty |t|\mathrm{d}\mu(t) < \infty$ and because all the moments exist. Consequently the right-hand side of (5.12) and hence the expression $\int_\gamma^\infty t f_z(t)\mathrm{d}\mu(t)$ is bounded at ∞. Hence from (5.11)-(5.12) we find that

$$\frac{\sigma_n(z)}{\varphi_n(z)} = \frac{1}{z} + A_n(z)\frac{1}{z^2}, \tag{5.13}$$

where $A_n(z)$ is bounded at ∞.

Similarly we have

$$\frac{1}{z-t} = \frac{1}{z} + \frac{1}{z^2}\frac{tz}{t-z} \tag{5.14}$$

and hence

$$S(z,\mu) = \frac{1}{z} + B(z)\cdot\frac{1}{z^2}, \tag{5.15}$$

where $B(z)$ is bounded at ∞.

From (5.13) and (5.15) we conclude that (5.7) is satisfied. □

Remark 5.3. In general, any rational function $P(z)/Q(z)$ where $\deg(P) < \deg(Q)$ interpolates $S(z,\mu)$ at ∞ in the sense that $S(z,\mu) - P(z)/Q(z) = C(z)z^{-1}$ when $\int_\gamma^\infty |t| \mathrm{d}\mu(t) < \infty$, where $C(z)$ is bounded at ∞. Thus one interpolation at ∞ by $\sigma_n(z)/\varphi_n(z)$ is guaranteed by definition. This is why an appropriate non-trivial interpolation condition at ∞ has the form (5.7).

6. Monotonicity of subsequences

We shall now present convergence results for the multi-point Padé approximants $\sigma_n(z)/\varphi_n(z)$. The content of the theorem below is established in Theorems 4.3, 4.5 and 5.3 of [11]. Related results on inclusion regions can be found in [2, pp. 244-246].

Theorem 6.1. *For $x \in (\alpha,\beta)$ the following results hold:*

A. *The sequence $\{\sigma_{2m}(x)/\varphi_{2m}(x)\}$ is decreasing to $S(x,\mu^{(0)})$, where $\mu^{(0)}$ is a certain solution of the SSMP on $\mathcal{L}$.*

B. *The sequence $\{\sigma_{2m+1}(x)/\varphi_{2m+1}(x)\}$ is increasing to $S(x,\mu^{(\infty)})$, where $\mu^{(\infty)}$ is a certain solution of the RSMP on $\mathcal{L}$.*

C. *For every solution ν of the RSMP on $\mathcal{L}\cdot\mathcal{L}$ we have*

$$S(x,\mu^{(\infty)}) \le S(x,\nu) \le S(x,\mu^{(\infty)}). \tag{6.1}$$

Since a measure is determined by the values of its Stieltjes transform on an arbitrary interval, we immediately obtain the following consequence of Theorem 6.1.

Corollary 6.2. *If $S(x,\mu^{(0)}) = S(x,\mu^{(\infty)})$ for $x \in (\alpha,\beta)$, then the RSMP on $\mathcal{L}\cdot\mathcal{L}$ is determinate.*

In other words: The RSMP on $\mathcal{L}\cdot\mathcal{L}$ is determinate if the sequence of $[n-1/n]$ multi-point Padé approximants of $S(z,\mu)$ at the tables above (or equivalently: the continued fraction (4.8)-(4.10) associated with the moment problem) converges for $z \in (\alpha,\beta)$.

References

[1] N.I. Akhiezer. *The classical moment problem and some related questions in analysis.* Hafner, New York, 1965.

[2] G.A. Baker Jr. *Essentials of Padé Approximants.* Academic Press, New York, 1975.

[3] G.A. Baker, Jr. and P.R. Graves-Morris. *Padé Approximants. Part II: Extensions and Applications*, volume 14 of *Encyclopedia of Mathematics and its Applications.* Addison-Wesley, Reading, MA, 1981.

[4] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions with poles on the unit circle. *J. Math. Anal. Appl.*, 182:221–243, 1994.

[5] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonality and boundary interpolation. In A.M. Cuyt, editor, *Nonlinear Numerical Methods and Rational Approximation II*, pages 37–48. Kluwer, 1994.

[6] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Recurrence relations for orthogonal functions. In S.C. Cooper and W.J. Thron, editors, *Continued Fractions and Orthogonal Functions*, volume 154 of *Lecture Notes in Pure and Appl. Math.*, pages 24–46. Marcel Dekker, 1994.

[7] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. A Favard theorem for rational functions with poles on the unit circle. *East J. Approx.*, 3:21–37, 1997.

[8] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. A rational moment problem on the unit circle. *Methods Appl. Anal.*, 4(3):283–310, 1997.

[9] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Continued fractions and orthogonal rational functions. In W.B. Jones and A.S. Ranga, editors, *Orthogonal Functions, Moment Theory and Continued Fractions: Theory and Applications*, volume 199 of *Lecture Notes in Pure and Applied Mathematics*, pages 69–100. Marcel Dekker, 1998.

[10] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. *Orthogonal rational functions*, volume 5 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, 1999.

[11] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. A rational Stieltjes problem. *Appl. Math. Comput.*, 1999. Submitted.

[12] A. Bultheel, P. González-Vera, and R. Orive. Quadrature on the half line and two-point Padé approximants to Stieltjes functions. Part I: Algebraic aspects. *J. Comput. Appl. Math.*, 65:57–72, 1995.

[13] T. Chihara. *An introduction to orthogonal polynomials.* Gordon and Breach Science Publishers, New York, 1978.

[14] M.A. Gallucci and W.B. Jones. Rational approximations corresponding to Newton series (Newton-Padé approximants). *J. Approx. Theory*, 17:366–392, 1976.

[15] A. Gonchar and G. López. On Markov's theorem for multipoint Padé approximants for functions of Stieltjes type. *Math. USSR-Sb.*, 105:512–524, 1978. Enlish translation. *Math. USSR-Sb*, 34 (1978) 449-459.

[16] P. González-Vera and O. Njåstad. Convergence of two-point Padé approximants to series of Stieltjes. *J. Comput. Appl. Math.*, 32:97–105, 1990.

[17] W.B. Gragg. Truncation error bounds for T-fractions. In E.W. Cheney, editor, *Approximation Theory III*, pages 455–460, New York, 1980. Academic Press.

[18] W.B. Jones, O. Njåstad, and W.J. Thron. Two-point Padé expansions for a family of analytic functions. *J. Comput. Appl. Math.*, 9:105–124, 1983.

[19] W.B. Jones and W. Thron. A postorior bound for truncated error of continued fractions. *SIAM J. Numer. Anal.*, 8:693–705, 1971.

[20] W.B. Jones and W.J. Thron. Two-point Padé tables and T-fractions. *Bull. Amer. Math. Soc.*, 83:388–390, 1977.

[21] W.B. Jones, W.J. Thron, and H. Waadeland. A strong Stieltjes moment problem. *Trans. Amer. Math. Soc.*, 206:503–528, 1980.

[22] J. Karlsson and B. von Sydow. The convergence of Padé approximants to series of Stieltjes. *Arch. Math.*, 14:42–53, 1976.

[23] G.L. Lopes [López-Lagomasino]. Conditions for convergence of multipoint Padé approximants for functions of Stieltjes type. *Math. USSR-Sb.*, 35:363–376, 1979.

[24] L. Lorentzen and H. Waadeland. *Continued fractions with applications*, volume 3 of *Studies in Computational Mathematics*. North-Holland, 1992.

[25] A. Magnus. On the structure of the two-point Padé table. In *Analytic theory of continued fractions*, volume 932 of *Lecture Notes in Math.*, pages 362–372, Berlin, 1982. Springer.

[26] J.H. McCabe. A formal extension of the Padé table to include two-point Padé quotients. *J. Inst. Math. Appl.*, 15:363–372, 1975.

[27] J.H. McCabe and J.A. Murphy. Continued fractions which correspond to power series expansions at two points. *J. Inst. Math. Appl.*, 17:233–247, 1976.

[28] O. Njåstad. Solutions of the Stieltjes moment problem. *Comm. Anal. Th. Continued Fractions*, 3:32–51, 1994.

[29] O. Njåstad. Solutions of the strong Stieltjes moment problem. *Methods Appl. Anal.*, 2:320–347, 1995.

[30] O. Njåstad. Solutions of the strong Hamburger moment problem. *J. Math. Anal. Appl.*, 197:227–248, 1996.

[31] O. Njåstad and W.J. Thron. The theory of sequences of orthogonal L-polynomials. In H. Waadeland and H. Wallin, editors, *Padé Approximants and Continued Fractions*, Det Kongelige Norske Videnskabers Selskab Skrifter (No. 1), pages 54–91, 1983.

[32] O. Perron. *Die Lehre von den Kettenbrüchen, vol. 2.* Teubner, Berlin, 1957.

[33] J.A. Shohat and J.D. Tamarkin. *The problem of moments*, volume 1 of *Math. Surveys*. Amer. Math. Soc., Providence, R.I., 1943.

[34] A. Sidi. Some aspects of the two-point Padé approximants. *J. Comput. Appl. Math.*, 6:9–17, 1980.

[35] T.J. Stieltjes. Recherches sur les fractions continues. *Ann. Fac. Sci. Toulouse*, 8:J.1–122, 9:A.1–47, 1894. English transl.: Oeuvres Complèts, Collected Papers, Springer Verlag, 1993, Vol. 2, 609–745.

[36] S. Tokarzewski, J. Bławzdziewicz, and I. Adrianov. Two-point Padé approximants for formal Stieltjes series. *Numer. Algorithms*, 8:313–328, 1994.

[37] W. Van Assche. The impact of Stieltjes' work on continued fractions and orthogonal polynomials. In *T.J. Stieltjes, Collected Papers*, pages 5–37. Stringer Verlag, 1993.

[38] H. Wallin. Rational interpolation to meromorphic functions. In M. de Bruin and H. van Rossum, editors, *Padé approximation and its applications*, volume 88 of *Lecture Notes in Math.*, pages 371–382. Springer Verlag, 1981.